Plan for today

1. Proofs by induction
True facts

- $1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$ for every integer $n \geq 1$

- If a sequence $u_n$ is given by
  
  $u_0 = 0 \quad u_{k+1} = 3u_k + 3^k$ for every integer $k \geq 0$

  then $u_n = n3^{n-1}$ for every integer $n \geq 0$

- $2n + 1 < 2^n$ for every integer $n \geq 3$

- $n^3 - 10n + 9$ is divisible by 3 for every integer $n \geq 1$

- $F_{k+1}^2 - F_k^2 = F_{k-1}F_{k+2}$ for every integer $k \geq 1$

Be a skeptic . . .

Try to disprove these statements by choosing some values of $n$ or $k$
Our first induction proof

We want to show that

\[ 1 + 2 + 3 + \cdots + n = \frac{n(n + 1)}{2} \]

for every integer \( n \geq 1 \).
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**Approach #1:**

*List all integers \( n \geq 1 \) and check that equation is true for each one*
Our first induction proof

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Approach #1:
List all integers \( n \geq 1 \) and check that equation is true for each one

- Try \( n = 1 \): . . .
- Try \( n = 2 \): . . .
- Try \( n = 3 \): . . .
Our first induction proof

We want to show that

\[ 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2} \]

for every integer \( n \geq 1 \).

**Approach #1:**
List all integers \( n \geq 1 \) and check that equation is true for each one

- Try \( n = 1 \): \( \ldots \)
- Try \( n = 2 \): \( \ldots \)
- Try \( n = 3 \): \( \ldots \)

Will this work?
Our first induction proof

We want to show that for every integer \( n \geq 1 \),

\[
1 + 2 + 3 + \cdots + n = \frac{n(n + 1)}{2}.
\]

**Approach #2:**
Recall that the integers greater than or equal to 1 have a recursive definition!

A. The smallest set such that
   - it includes 0, and
   - for each number \( n \in X \), \( n + 1 \in X \) as well.

B. The smallest set such that
   - it includes 0, and
   - for each number \( n \in X \), \( n + 2 \in X \) as well.

C. The smallest set such that
   - it includes 1, and
   - for each number \( n \in X \), \( n + 1 \in X \) as well.

D. The smallest set such that
   - it includes 1, and
   - for each number \( n \in X \), \( n + 2 \in X \) as well.

E. None of the above.
Our first induction proof

We want to show that for every integer $n \geq 1$,
$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}.$$  

Using the recursive definition, we can cover all elements of the set by

- **Base value:** the set includes 1
- **Induction rule:** if the set includes $n$ then it also includes $n + 1$. 
Our first induction proof

We want to show that for every integer $n \geq 1$,
\[1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}.
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1
Our first induction proof

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Our first induction proof

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1 2 3 4 5
Our first induction proof

We want to show that for every integer $n \geq 1$,

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Using the recursive definition, we can cover all elements of the set by

- **Base value**: the set includes 1
- **Induction rule**: if the set includes $n$ then it also includes $n + 1$.

\[
\begin{array}{cccccc}
1 & 2 & 3 & 4 & 5 & 6 \ldots \\
\end{array}
\]
Our first induction proof

We want to show that for every integer $n \geq 1$, $1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$.

The Base Case:

$$1 + 2 + 3 + \cdots + n \stackrel{?}{=} \frac{n(n+1)}{2}.$$
Our first induction proof

We want to show that for every integer $n \geq 1$, $1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$.

The Base Case:

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}.$$
Our first induction proof

We want to show that for every integer $n \geq 1$, $1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$.

The Base Case:

$$1 + 2 + 3 + \cdots + 1 \mathrel{?} \frac{1(1+1)}{2}.$$
Our first induction proof

We want to show that for every integer $n \geq 1$, $1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$.

The Base Case:

$$1 \stackrel{?}{=} \frac{1(1 + 1)}{2}$$

Is this equation true?

A. Yes

B. No

C. ???
Our first induction proof

We want to show that for every integer \( n \geq 1 \),
\[
1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}.
\]

The Induction Step (warmup):

Suppose we proved that \( \text{EQUATION}_{n=17} \) is true. Can we use this to prove that \( \text{EQUATION}_{n=18} \) is true?
Our first induction proof

We want to show that for every integer \( n \geq 1 \),
\[
1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}.
\]

The Induction Step (warmup):

Suppose we proved that \( \text{EQUATION}_{n=17} \) is true. Can we use this to prove that \( \text{EQUATION}_{n=18} \) is true?

What is LeftHandSide for each of \( n = 17 \) and \( n = 18 \)?

A. 17 and 18
B. 1 and 2
C. \( 1 + 2 + 3 + \cdots + 17 \) and \( 2 + 3 + \cdots + 18 \)
D. \( 1 + 2 + 3 + \cdots + 17 \) and \( 1 + 2 + 3 + \cdots + 18 \)
E. None of the above.
Our first induction proof

We want to show that for every integer \( n \geq 1 \),
\[ 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}. \]

The Induction Step (warmup):

Suppose we proved that \( EQUATION_{n=17} \) is true. Can we use this to prove that \( EQUATION_{n=18} \) is true?

What is RightHandSide for each of \( n = 17 \) and \( n = 18 \)?

A. 17 and 18
B. \( \frac{17}{2} \) and \( \frac{18}{2} \)
C. \( \frac{18}{2} \) and \( \frac{19}{2} \)
D. \( \frac{17 \cdot 18}{2} \) and \( \frac{18 \cdot 19}{2} \)
E. None of the above.
Our first induction proof

We want to show that for every integer $n \geq 1$, \[1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}\].

The Induction Step (warmup):

\[
\begin{array}{c|c|c}
& n = 17 & n = 18 \\
LHS & 1 + 2 + 3 + \cdots + 17 & 1 + 2 + 3 + \cdots + 18 \\
RHS & \frac{17 \cdot 18}{2} & \frac{18 \cdot 19}{2}
\end{array}
\]
Our first induction proof

We want to show that for every integer $n \geq 1$, $1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$.

The Induction Step (warmup):

<table>
<thead>
<tr>
<th></th>
<th>$n = 17$</th>
<th>$n = 18$</th>
</tr>
</thead>
<tbody>
<tr>
<td>LHS</td>
<td>$1 + 2 + 3 + \cdots + 17$</td>
<td>$1 + 2 + 3 + \cdots + 18$</td>
</tr>
<tr>
<td>RHS</td>
<td>$\frac{17 \cdot 18}{2}$</td>
<td>$\frac{18 \cdot 19}{2}$</td>
</tr>
</tbody>
</table>

Know are EQUAL

Want to show EQUAL
Our first induction proof

We want to show that for every integer $n \geq 1$, $1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$.

There’s nothing special about the numbers 17 and 18!

**The Induction Step:**
- Let $n$ be an arbitrary integer $\geq 1$.
- Assume $EQUATION_n$
- Deduce $EQUATION_{n+1}$
Observations about induction

- Why is the base case separate?
- How do we choose the base case?
- What does arbitrary mean?

Schematic for verifying an induction proof
In previous example, we thought about the sum

$$1 + 2 + 3 + \cdots + n.$$ 

Shorthand:

$$\sum_{i=1}^{n} i$$
What is \[ \sum_{i=1}^{4}(i^2 - i) \]?

A. 20
B. −10
C. 30
D. 0
E. None of the above.
Recursive function

\[ f : \mathbb{N} \to \mathbb{N} \]

- \( f(0) = \) __________
- \( f(n) = \ldots \text{uses } f(n - 1) \ldots \)

We’ve seen this before:

- Sequences defined by recursion
- Factorial function
Sum function

\[ \text{sum} : \mathbb{N} \rightarrow \mathbb{N} \]

\[ \text{sum}(n) = 0 + 1 + 2 + \cdots + n \]

- \( \text{sum}(0) = \) 
- \( \text{sum}(n) = \ldots \text{uses} \ \text{sum}(n - 1) \ldots \)
Sum function

\[ sum : \mathbb{N} \rightarrow \mathbb{N} \]

\[ sum(n) = 0 + 1 + 2 + \cdots + n \]

- \( sum(0) = \ldots \)

What is \( sum(0) \)?

- \( sum(n) = \ldots \text{uses } sum(n - 1) \ldots \)
Sum function

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\[ \text{sum}(n) = 0 + 1 + 2 + \cdots + n \]

- \( \text{sum}(0) = \) ______

What is \( \text{sum}(0) \)?

- \( \text{sum}(n) = \ldots \text{uses} \ \text{sum}(n-1) \ldots \)

What is \( \text{sum}(n) \)?
More sum notation

Evaluate

\[ \sum_{i=1}^{10} \sqrt{i} \]

\( i \) is a square

A. 55
B. 22.468…
C. 14
D. 6