Plan for today

1. More proofs by induction
2. Prime numbers
3. Divisibility
4. Strong induction
Induction

Which of the following statements do you agree with?

A. Proofs by induction always work for proving identities about sequences.
B. Proofs by induction can only be used to prove two quantities are equal.
C. Proofs by induction depend on a recursive definition of the underlying set.
D. None of the above.
Goal: $2n + 1 < 2^n$ for every integer $n \geq 3$
Proving an inequality

**Goal:** \(2n + 1 < 2^n\) for every integer \(n \geq 3\)

- **Base case:**
- **Induction step:** declare variables!
  - Assumption: previous term
  - Subgoal: next term
Goal: \(2n + 1 < 2^n\) for every integer \(n \geq 3\)

An alternate strategy: Growth rates
More than one road to Timbuktu

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*An alternate strategy:* Growth rates

- Idea: exponentials grow faster than linear functions
More than one road to Timbuktu

Goal: \( 2n + 1 < 2^n \) for every integer \( n \geq 3 \)

An alternate strategy: Growth rates

- Idea: exponentials eventually grow faster than linear functions
- but be careful ...

\[ 100n < 2^n \]
Goal: $n^3 - 10n + 9$ is divisible by 3 for every integer $n \geq 1$
Proving a property

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More than one road to Timbuktu

Goal: \( n^3 - 10n + 9 \) is divisible by 3 for every integer \( n \geq 1 \)

An alternate strategy:
Group integers according to their remainder mod 3
More than one road to Timbuktu

Goal: \( n^3 - 10n + 9 \) is divisible by 3 for every integer \( n \geq 1 \)

Each integer \( n \geq 1 \) falls into (exactly) one of three categories:

- \( n \mod 3 \equiv 0 \)
- \( n \mod 3 \equiv 1 \)
- \( n \mod 3 \equiv 2 \)
Proving statement about Fibonacci numbers

Fibonacci sequence:

\[ F_0 = 0, \quad F_1 = 1, \quad \text{and} \quad F_n = F_{n-1} + F_{n-2} \text{ for } n \geq 2 \]

Want to prove:

\[ F_{k+1}^2 - F_k^2 = F_{k-1}F_{k+2} \]

for every integer \( k \geq 1 \).
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*Hint:* Rewrite equation as \( F_{k+1}^2 = F_k^2 + F_{k-1}F_{k+2} \)
Proving statement about Fibonacci numbers

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\[ F_{k+1}^2 = F_k^2 + F_{k-1}F_k \]

for every integer \( k \geq 1 \).
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Goal:

\[ \sum_{i=0}^{n} F_i = F_{n+2} - 1 \]

for every integer \( n \geq 0 \).

Do you believe this identity?
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for every integer \( n \geq 0 \).

Do you believe this identity?

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